

NON-ARCHIMEDEAN CONNECTED JULIA SETS WITH BRANCHING

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ABSTRACT. We construct the first examples of rational functions defined over a non-archimedean field with a certain dynamical property: the Julia set in the Berkovich projective line is connected but not contained in a line segment. We also show how to compute the measure-theoretic and topological entropy of such maps. In particular, we give an example for which the measure-theoretic entropy is strictly smaller than the topological entropy, thus answering a question of Favre and Rivera-Letelier.

Let \mathbb{C}_v be an algebraically closed field equipped with a non-archimedean absolute value $|\cdot|_v$. That is, $|\cdot|_v : \mathbb{C}_v \rightarrow [0, \infty)$ with $|x|_v = 0$ if and only if $x = 0$, with $|xy|_v = |x|_v|y|_v$, and satisfying the non-archimedean triangle inequality

$$|x + y|_v \leq \max\{|x|_v, |y|_v\}.$$

We will assume that \mathbb{C}_v is complete with respect to $|\cdot|_v$; that is, all $|\cdot|_v$ -Cauchy sequences converge. The *ring of integers* \mathcal{O} of \mathbb{C}_v is the closed unit disk $\mathcal{O} := \{x \in \mathbb{C}_v : |x|_v \leq 1\}$, which forms a subring of \mathbb{C}_v with unique maximal ideal $\mathcal{M} := \{x \in \mathbb{C}_v : |x|_v < 1\}$. The *residue field* k of \mathbb{C}_v is the quotient \mathcal{O}/\mathcal{M} .

The *degree* of a rational function $\phi \in \mathbb{C}_v(z)$ is $\deg \phi := \max\{\deg f, \deg g\}$, where $\phi = f/g$ and $f, g \in \mathbb{C}_v[z]$ are relatively prime polynomials. For $n \geq 0$, we denote the n -th iterate of ϕ under composition by ϕ^n ; that is, $\phi^0(z) = z$, and $\phi^{n+1} = \phi \circ \phi^n$.

The rational function ϕ acts naturally on the Berkovich projective line $\mathbb{P}_{\text{Ber}}^1$. We will discuss $\mathbb{P}_{\text{Ber}}^1$ in more detail in Section 1; for the moment, we note only that $\mathbb{P}_{\text{Ber}}^1$ is a certain path-connected compactification of $\mathbb{P}^1(\mathbb{C}_v)$, and that many of the extra points correspond to closed disks in \mathbb{C}_v . We also note that the set $\mathbb{H}_{\text{Ber}} := \mathbb{P}_{\text{Ber}}^1 \setminus \mathbb{P}^1(\mathbb{C}_v)$ has a natural metric $d_{\mathbb{H}}$, although the metric topology on \mathbb{H}_{Ber} is stronger than the topology inherited from $\mathbb{P}_{\text{Ber}}^1$. We call \mathbb{H}_{Ber} *hyperbolic space*, and $d_{\mathbb{H}}$ the *hyperbolic metric*.

The dynamical action of ϕ partitions $\mathbb{P}_{\text{Ber}}^1$ into two invariant subsets: the (Berkovich) Julia set \mathcal{J}_{ϕ} , which is the closed subset on which ϕ^n acts chaotically, and the (Berkovich) Fatou set \mathcal{F}_{ϕ} , which is the (open) complement of \mathcal{J}_{ϕ} . There are numerous examples of Berkovich Julia sets in the literature; see, for example, [2, 8, 9, 12, 18, 20, 21, 22]. However, in all these examples, the Julia set is either a single point, a line segment, or a disconnected set (in which case it necessarily has infinitely many connected components). In this paper, we give the first examples of Berkovich Julia sets that are connected but are not contained in a line segment.

The main engine we use to produce our examples is the following theorem. To state it, we note that a *finite tree* $\Gamma \subseteq \mathbb{P}_{\text{Ber}}^1$ is exactly what it sounds like: a subset of $\mathbb{P}_{\text{Ber}}^1$ homeomorphic

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to a finite tree. Similarly, an *interval* $I \subseteq \mathbb{P}_{\text{Ber}}^1$ is a subset of $\mathbb{P}_{\text{Ber}}^1$ homeomorphic to an interval in \mathbb{R} . See Section 1 for details.

Theorem A. *Let $\phi \in \mathbb{C}_v(z)$ be a rational function of degree at least 2 with Berkovich Julia set \mathcal{J}_ϕ . Let $\Gamma \subseteq \mathbb{H}_{\text{Ber}}$ be a finite tree, let $I \subseteq \Gamma$ be a compact interval, and let $x_0 \in I$. Suppose that I can be written as a union of intervals $I = I_1 \cup \cdots \cup I_m$ such that*

- (a) $x_0 \in \mathcal{J}_\phi$,
- (b) $\phi^{-1}(x_0) \subseteq \Gamma$,
- (c) $\bigcup_{n \geq 0} \phi^{-n}(I) \supseteq \Gamma$, and
- (d) *for each $i = 1, \dots, m$, there are integers $b_i \geq 1$ and $c_i \geq 2$ such that*
 - ϕ^{b_i} maps I_i onto I , with
 - $d_{\mathbb{H}}(\phi^{b_i}(x), \phi^{b_i}(y)) = c_i d_{\mathbb{H}}(x, y)$ for all $x, y \in I_i$.

Then \mathcal{J}_ϕ is connected and contains Γ .

Moreover, if Γ is not contained in an interval, then \mathcal{J}_ϕ has a dense subset of branch points, i.e., points $x \in \mathcal{J}_\phi$ for which $\mathcal{J}_\phi \setminus \{x\}$ has at least three connected components.

Julia sets satisfying the full conditions of Theorem A can be considered dendrites, much like the complex Julia set of $\phi(z) = z^2 + i \in \mathbb{C}(z)$. In fact, however, there are maps with connected Julia set having a dense set of points of infinite branching — that is, points $x \in \mathcal{J}_\phi$ for which $\mathcal{J}_\phi \setminus \{x\}$ has infinitely many connected components. In Theorem 3.3, we will give a simple sufficient condition for such infinite branching to occur. These dendritic Julia sets are rather more complicated than those like the complex Julia set of $z^2 + i$, for which there is only finite branching at each branch point.

We will also be interested the entropy of a map $\phi \in \mathbb{C}_v(z)$ acting on $\mathbb{P}_{\text{Ber}}^1$. In particular, in [12], Favre and Rivera-Letelier discuss the measure-theoretic entropy $h_\mu(\phi)$ and topological entropy $h_{\text{top}}(\phi)$ of ϕ acting on $\mathbb{P}_{\text{Ber}}^1$. (Here, $\mu = \mu_\phi$ is a certain natural measure on $\mathbb{P}_{\text{Ber}}^1$ supported on the Julia set \mathcal{J}_ϕ ; see Sections 2.1 and 2.2 for details.) They prove that

$$0 \leq h_\mu(\phi) \leq h_{\text{top}}(\phi) \leq \log \deg(\phi),$$

and they give examples, some with Julia set a line segment, where the inequalities are all strict. Since their example maps have fairly large degree, in [12, Question 3], they ask, among other things, whether there exist rational functions $\phi \in \mathbb{C}_v(z)$ of degree at most 9 where the inequalities are strict and the Julia set is connected.

We answer this question positively when the residue characteristic p of \mathbb{C}_v is 3. (In a separate paper, [4], the second author will present an example of degree 4, in residue characteristic $p = 2$, for which the inequalities are strict and the Julia set is connected.) In particular, for $p = 3$ and $a \in \mathbb{C}_v^\times$ satisfying $|3|_v \leq |a|_v < 1$, we will show that

$$\phi(z) = \frac{az^6 + 1}{az^6 + z^3 - z} = 1 + \frac{-z^3 + z + 1}{az^6 + z^3 - z}$$

has connected Julia set and entropies

$$h_\mu(\phi) = \log 2 + \frac{5}{11} \log 3 \approx 1.1925 \dots, \quad \text{and} \quad h_{\text{top}}(\phi) = \log \lambda \approx 1.3496 \dots,$$

where $\lambda \approx 3.8558 \dots$ is the largest real root of the polynomial $t^3 - 4t^2 - t + 6$. To our knowledge, ϕ is the first rational function to appear in the literature which acts on $\mathbb{P}_{\text{Ber}}^1$ with topological entropy *not* the logarithm of an integer.

To compute these entropies, we will use the following result.

Theorem B. *Let $\phi \in \mathbb{C}_v(z)$ be a rational function of degree $d \geq 2$, with Julia set \mathcal{J}_ϕ and invariant measure μ . Suppose that \mathcal{J}_ϕ is connected and of finite diameter with respect to the hyperbolic metric $d_{\mathbb{H}}$. Let \mathcal{P} be a countable partition of \mathcal{J}_ϕ such that for every $U \in \mathcal{P}$,*

- *U is path-connected and is the union of an open set and a countable set;*
- *there is a set $S_U \subseteq \mathcal{P}$ such that ϕ maps U bijectively onto $\bigcup_{V \in S_U} V$; and*
- *for all $x \neq y \in U$, there is an integer $n \geq 0$ such that either $\phi^n(x)$ and $\phi^n(y)$ belong to different elements of \mathcal{P} , or else $d_{\mathbb{H}}(\phi^n(x), \phi^n(y)) \geq 2d_{\mathbb{H}}(x, y)$.*

Then

- (a) *\mathcal{P} is a one-sided generator for $\phi : \mathcal{J}_\phi \rightarrow \mathcal{J}_\phi$ of finite entropy.*
- (b) $h_\mu(\phi) = \int_{\mathbb{P}_{\text{Ber}}^1} \log \left(\frac{d}{\deg_x(\phi)} \right) d\mu(x).$
- (c) *Let $\mathcal{A} \subseteq \mathcal{P}$ consist of those elements of \mathcal{P} that are uncountable sets, and let T be the one-sided topological Markov shift on the symbol space \mathcal{A} , where symbol $U \in \mathcal{A}$ can be followed by symbol $V \in \mathcal{A}$ if and only if $V \in S_U$. Then $h_{\text{top}}(\phi) = h_{\text{Gur}}(T)$, where $h_{\text{Gur}}(T)$ is the Gurevich entropy of T .*

The existence of a one-sided generator of finite entropy is a technical condition which enables the explicit computation of the entropy; see the proof of Theorem B in Section 4. Meanwhile, the Gurevich entropy $h_{\text{Gur}}(T)$ mentioned at the end of Theorem B is the analog of the topological entropy for a countable-state Markov shift T ; see Section 2.4.

The outline of the paper is as follows. In Section 1 we discuss the basic properties of the Berkovich projective line $\mathbb{P}_{\text{Ber}}^1$. In Section 2, we give some relevant background on the theory of dynamics on $\mathbb{P}_{\text{Ber}}^1$; we also include some discussion of entropy, especially Gurevich entropy. We prove Theorem A in Section 3, and Theorem B in Section 4. Finally, in Section 5, we present our degree six example.

1. BACKGROUND ON THE BERKOVICH PROJECTIVE LINE

1.1. Berkovich points and disks. The Berkovich projective line $\mathbb{P}_{\text{Ber}}^1$ over \mathbb{C}_v is a certain compact Hausdorff topological space containing $\mathbb{P}^1(\mathbb{C}_v)$ as a subspace. The precise definition, involving multiplicative seminorms on \mathbb{C}_v -algebras, is tangential to this paper; the interested reader may consult Berkovich's monograph [5], the detailed exposition in [2], or the summary in [3], for example. We give only an overview here, without proofs.

There are four types of points in $\mathbb{P}_{\text{Ber}}^1$. Type I points are simply the points of $\mathbb{P}^1(\mathbb{C}_v)$. Points of type II and III correspond to closed disks in \mathbb{C}_v . Specifically, for $a \in \mathbb{C}_v$ and $r > 0$, the closed disk $\overline{D}(a, r) := \{z \in \mathbb{C}_v : |z - a|_v \leq r\}$ corresponds to a unique point, which we shall denote $\zeta(a, r)$, in $\mathbb{P}_{\text{Ber}}^1$. If $r \in |\mathbb{C}_v^\times|_v$, the point $\zeta(a, r)$ is of type II; otherwise, $\zeta(a, r)$ is of type III. The type II point $\zeta(0, 1)$ is known as the *Gauss point*. The remaining points of $\mathbb{P}_{\text{Ber}}^1$, known as the type IV points, correspond to equivalence classes of descending chains $D_1 \supsetneq D_2 \supsetneq \cdots$ of disks in \mathbb{C}_v with empty intersection. Type IV points will not be important in this paper, though.

The absolute value $|\cdot|_v$ extends to a function $|\cdot|_v : \mathbb{P}_{\text{Ber}}^1 \setminus \{\infty\} \rightarrow [0, \infty)$, where in particular $|\zeta(a, r)|_v := \max\{|a|_v, r\}$.

The space $\mathbb{P}_{\text{Ber}}^1$ is equipped with a topology, known as either the *Gel'fand topology* or the *weak topology*. The Gel'fand topology restricted to $\mathbb{P}^1(\mathbb{C}_v)$ coincides with the usual topology induced by $|\cdot|_v$. Unlike $\mathbb{P}^1(\mathbb{C}_v)$, however, $\mathbb{P}_{\text{Ber}}^1$ is compact (and still Hausdorff).

Given $a \in \mathbb{C}_v$ and $r > 0$, $\overline{\mathcal{D}}(a, r) \subseteq \mathbb{P}_{\text{Ber}}^1$ will denote a certain closed subset whose type I points are those in $\overline{\mathcal{D}}(a, r)$, and whose type II and III points are of the form $\zeta(b, s)$ with $\overline{\mathcal{D}}(b, s) \subseteq \overline{\mathcal{D}}(a, r)$. Similarly, $\mathcal{D}(a, r) \subseteq \mathbb{P}_{\text{Ber}}^1$ is a certain open subset whose type I points are those in $\mathcal{D}(a, r) \subseteq \mathbb{C}_v$, and whose type II and III points are of the form $\zeta(b, s)$ with $\overline{\mathcal{D}}(b, s) \subseteq \mathcal{D}(a, r)$ other than $\zeta(a, r)$ itself.

More generally, a *closed Berkovich disk* is a subset of $\mathbb{P}_{\text{Ber}}^1$ of the form either $\mathcal{D}(a, r)$ or $\mathbb{P}_{\text{Ber}}^1 \setminus \overline{\mathcal{D}}(a, r)$. Similarly, an *open Berkovich disk* is a subset of $\mathbb{P}_{\text{Ber}}^1$ of the form either $\overline{\mathcal{D}}(a, r)$ or $\mathbb{P}_{\text{Ber}}^1 \setminus \mathcal{D}(a, r)$. For each of these four possibilities, the boundary of the Berkovich disk consists of the single point $\zeta(a, r)$. Meanwhile, a *closed* (respectively, *open*) *connected affinoid* U is a nonempty finite intersection of closed (respectively, open) Berkovich disks. If none of the disks in the intersection contains any other, then the boundary of U consists of the (finitely many) boundary points of the disks in the intersection.

1.2. Path-connectedness and the hyperbolic metric. The space $\mathbb{P}_{\text{Ber}}^1$ is uniquely path-connected. That is, for any $x \neq y \in \mathbb{P}_{\text{Ber}}^1$, there is a unique subspace $I \subseteq \mathbb{P}_{\text{Ber}}^1$ homeomorphic to the unit interval $[0, 1] \subseteq \mathbb{R}$, where $x, y \in I$, and the homeomorphism takes x to 0 and y to 1. The set I is called an *interval* in $\mathbb{P}_{\text{Ber}}^1$, and it is denoted $[x, y]$ or $[y, x]$. The points in $[x, y]$ are said to *lie between* x and y . For example, if $x = \zeta(a, r)$ and $y = \zeta(a, s)$ with $s > r > 0$, then the interval $[x, y]$ consists of all points $\zeta(a, t)$ with $r \leq t \leq s$. $\mathbb{P}_{\text{Ber}}^1$ is also locally path-connected. Thus, any connected open subset is, like $\mathbb{P}_{\text{Ber}}^1$ itself, uniquely path-connected.

Given a set of points $S \subseteq \mathbb{P}_{\text{Ber}}^1$, the *convex hull* of S is the set $\Gamma := \bigcup_{x, y \in S} [x, y]$ consisting of all points lying between two points of S . If S is finite and nonempty, then the convex hull Γ is a finite tree. Conversely, any finite tree $\Gamma \subseteq \mathbb{P}_{\text{Ber}}^1$ is the convex hull of some nonempty finite set $S \subseteq \mathbb{P}_{\text{Ber}}^1$.

The unique path-connectedness endows $\mathbb{P}_{\text{Ber}}^1$ with a tree-like structure; however, unlike a finite tree, it has points of infinite branching, and such points are dense in any interval $[x, y]$ with $x \neq y$. In fact, these infinitely-branched points are precisely the type II points, because for any type II point $x = \zeta(a, r)$, the complement $\mathbb{P}_{\text{Ber}}^1 \setminus \{x\}$ consists of infinitely many connected components, called the *residue classes* at x . Each of these components is an open Berkovich disk: one is $\mathbb{P}_{\text{Ber}}^1 \setminus \overline{\mathcal{D}}(a, r)$, while the rest are the infinitely many disks $\mathcal{D}(b, r) \subseteq \mathbb{P}^1(\overline{K})$ with $b \in \overline{\mathcal{D}}(a, r)$. Thus, the residue classes at x are in one-to-one correspondence with the points of $\mathbb{P}^1(k)$, where k is the residue field of \mathbb{C}_v .

On the other hand, if $\zeta(a, r) \in \mathbb{P}_{\text{Ber}}^1$ is of type III, then $\mathbb{P}_{\text{Ber}}^1 \setminus \{\zeta(a, r)\}$ has two components: the open Berkovich disks $\mathcal{D}(a, r)$ and $\mathbb{P}_{\text{Ber}}^1 \setminus \overline{\mathcal{D}}(a, r)$. In addition, if $x \in \mathbb{P}_{\text{Ber}}^1$ is either of type I or type IV, then $\mathbb{P}_{\text{Ber}}^1 \setminus \{x\}$ is still connected.

The set $\mathbb{H}_{\text{Ber}} := \mathbb{P}_{\text{Ber}}^1 \setminus \mathbb{P}^1(\mathbb{C}_v)$ is called *hyperbolic space* and admits a metric $d_{\mathbb{H}}$, which can be easily described on the type II and III points. Given $a \in \mathbb{C}_v$ and $r \geq s > 0$, we have

$$d_{\mathbb{H}}(\zeta(a, r), \zeta(a, s)) := \log r - \log s.$$

The hyperbolic distance between two points $\zeta(a, r)$ and $\zeta(b, s)$ is then the sum of the distances from each point to $\zeta(c, t)$, where $\overline{\mathcal{D}}(c, t)$ is the smallest closed disk in \mathbb{C}_v containing both $\overline{\mathcal{D}}(a, r)$ and $\overline{\mathcal{D}}(b, s)$. The reader should be warned, however, that the $d_{\mathbb{H}}$ -metric topology on \mathbb{H}_{Ber} is strictly stronger than Gel'fand topology, and hence the former is known as the *strong topology*. However, if $\Gamma \subseteq \mathbb{H}_{\text{Ber}}$ is a finite tree, then the Gel'fand and strong topologies each induce the same subspace topology on Γ .

1.3. Rational functions on $\mathbb{P}_{\text{Ber}}^1$. Any nonconstant rational function $\phi \in \mathbb{C}_v(z)$ extends uniquely to a continuous function from $\mathbb{P}_{\text{Ber}}^1$ to $\mathbb{P}_{\text{Ber}}^1$. This extension, which we also denote ϕ , is in fact an open map. For each point $x \in \mathbb{P}_{\text{Ber}}^1$, the image $\phi(x)$ is a point of the same type. In addition, ϕ has a local degree $\deg_x(\phi)$ at x . The local degree is an integer between 1 and $\deg \phi$; in fact, for any $y \in \mathbb{P}_{\text{Ber}}^1$, we have

$$(1.1) \quad \sum_{x \in \phi^{-1}(y)} \deg_x(\phi) = \deg \phi.$$

That is, every point of $\mathbb{P}_{\text{Ber}}^1$ has exactly $\deg \phi$ preimages, counted with multiplicity. Rather than formally defining ϕ and $\deg_x(\phi)$ on $\mathbb{P}_{\text{Ber}}^1$, we note the following special cases.

- (a) If $D(a, r) \subseteq \mathbb{C}_v$ is an open disk containing no poles of ϕ , then the image $\phi(D(a, r))$ is also an open disk $D(b, s)$, and we have $\phi(\zeta(a, r)) = \zeta(b, s)$. If there are no poles in the closed disk $\overline{D}(a, r)$, then $\phi : \overline{D}(a, r) \rightarrow \overline{D}(b, s)$ is everywhere m -to-1, counting multiplicity, for some $m \geq 1$; in that case, $\deg_{\zeta(a, r)}(\phi) = m$.
- (b) Writing $\phi = f/g$ with $f, g \in \mathcal{O}[z]$ and at least one coefficient of f or g in \mathcal{O}^\times , define

$$\overline{\phi} := \overline{f}/\overline{g} \in k(z) \cup \{\infty\}.$$

We have $\phi(\zeta(0, 1)) = \zeta(0, 1)$ if and only if $\overline{\phi}$ is not constant. In that case, $\deg_{\zeta(0, 1)}(\phi) = \deg(\overline{\phi})$.

- (c) Given $\psi \in \mathbb{C}_v(z)$ and $w \in \mathbb{C}_v$, we will write

$$(1.2) \quad \phi(w) \approx \psi(w) \quad \text{if} \quad |\phi(w) - \psi(w)|_v < |\phi(w)|_v.$$

Fix $a \in \mathbb{C}_v$ and $r > 0$. Suppose there is some $\psi \in \mathbb{C}_v(z)$ and $b \in \mathbb{C}_v$ such that

- $\phi(w) - b \approx \psi(w)$ for all $w \in \mathbb{C}_v$ with $|w - a|_v < r$, and
- $\psi(\zeta(a, r)) = \zeta(b, s)$ for some $s > 0$.

Then $\phi(\zeta(a, r)) = \zeta(b, s)$, and $\deg_{\zeta(a, r)}(\phi) = \deg_{\zeta(a, r)}(\psi)$.

If $U \subseteq \mathbb{P}_{\text{Ber}}^1$ is connected and $\deg_x(\phi) = n$ is constant on U , then

$$d_{\mathbb{H}}(\phi(x), \phi(y)) = nd_{\mathbb{H}}(x, y) \quad \text{for all } x, y \in U.$$

2. DYNAMICS ON THE BERKOVICH PROJECTIVE LINE

2.1. Berkovich Julia sets. Just as in complex dynamics, a rational function $\phi \in \mathbb{C}_v(z)$ has an associated Julia set. As in Section 1, we will only give definitions and state properties without proofs. For further details, see, for example, [3, Section 6.4], [2, Section 10.5], or [12, Section 2.3].

Definition 2.1. Let $\phi \in \mathbb{C}_v(z)$ be a rational function of degree $d \geq 2$. The (*Berkovich*) *Julia set* of ϕ is the set $\mathcal{J}_\phi \subseteq \mathbb{P}_{\text{Ber}}^1$ consisting of those points $x \in \mathbb{P}_{\text{Ber}}^1$ with the property that for any open set $U \subseteq \mathbb{P}_{\text{Ber}}^1$ containing x ,

$$\mathbb{P}_{\text{Ber}}^1 \setminus \left[\bigcup_{n \geq 0} \phi^{-n}(U) \right] \quad \text{is finite.}$$

The Berkovich Julia set of $\phi \in \mathbb{C}_v(z)$ has a number of properties familiar to complex dynamicists. First, it is closed in $\mathbb{P}_{\text{Ber}}^1$ and hence compact. Second, it is invariant under ϕ , in the sense that $\phi^{-1}(\mathcal{J}_\phi) = \phi(\mathcal{J}_\phi) = \mathcal{J}_\phi$. Third, for any $x \in \mathcal{J}_\phi$, the backward orbit

$$\mathcal{O}_\phi^-(x) := \bigcup_{n \geq 0} \phi^{-n}(x)$$

is a dense subset of \mathcal{J}_ϕ . Fourth, if $x = \zeta(a, r)$ is a repelling periodic point — i.e., if $\phi^n(x) = x$ and $\deg_x(\phi^n) \geq 2$ for some $n \geq 1$ — then $x \in \mathcal{J}_\phi$.

Meanwhile, it is possible that the \mathcal{J}_ϕ consists of a single point x , which is necessarily a fixed point of type II satisfying $\deg_x(\phi) = \deg \phi$. In that case, ϕ is said to have *potentially good reduction*. However, we will be mainly concerned with maps ϕ that do *not* have potentially good reduction, i.e., for which \mathcal{J}_ϕ consists of more than one point. In that case, \mathcal{J}_ϕ is a perfect set and in particular is uncountable.

We note two other useful properties of the Julia set. First, for any open set U intersecting \mathcal{J}_ϕ , some iterate $\phi^n(U)$ contains \mathcal{J}_ϕ , by [2, Theorem 10.56(B')]. Second, \mathcal{J}_ϕ is a separable metrizable space and hence is second countable; in fact, it has a basis consisting of countably many open connected affinoids intersected with \mathcal{J}_ϕ . (This is true by [10, Lemma 7.15], because, as we will see in Section 2.2, \mathcal{J}_ϕ is the support of a Borel measure.) It follows that \mathcal{J}_ϕ can intersect only countably many residue classes of any point $x \in \mathbb{P}_{\text{Ber}}^1$.

2.2. The invariant measure. Again in parallel with complex dynamics, given a rational function $\phi \in \mathbb{C}_v(z)$ of degree $d \geq 2$, there is a naturally associated Borel probability measure $\mu = \mu_\phi$ on $\mathbb{P}_{\text{Ber}}^1$. (The construction of this measure, via potential theory on $\mathbb{P}_{\text{Ber}}^1$, first appeared in [1, 11]; see also [2, Section 10.1] and [12].) The measure μ is invariant with respect to ϕ , meaning that $\mu(U) = \mu(\phi^{-1}(U))$ for any Borel set $U \subseteq \mathbb{P}_{\text{Ber}}^1$. According to [12, Théorème A], it is also mixing and hence ergodic, so that any Borel set U with $\phi^{-1}(U) = U$ has either $\mu(U) = 0$ or $\mu(U) = 1$. By the same theorem, its support $\text{Supp } \mu$ is precisely \mathcal{J}_ϕ ; see also [2, Sections 10.1, 10.5].

In the context of ergodic theory, the *Jacobian* of $\phi : \mathbb{P}_{\text{Ber}}^1 \rightarrow \mathbb{P}_{\text{Ber}}^1$ is a nonnegative-valued function such that for every Borel set $U \subseteq \mathbb{P}_{\text{Ber}}^1$ on which ϕ is injective, we have $\mu(\phi(U)) = \int_U \text{Jac}_\phi(x) d\mu(x)$. In [12, Lemme 4.4(2)], Favre and Rivera-Letelier proved that any rational function $\phi \in \mathbb{C}_v(z)$ of degree $d \geq 2$ has a Jacobian, given by

$$(2.1) \quad \text{Jac}_\phi(x) = \frac{d}{\deg_x(\phi)}.$$

2.3. Entropy. Let X be a topological space and $f : X \rightarrow X$ a continuous function. If μ is a Borel probability measure on X and \mathcal{P} is a finite partition of X into measurable pieces, the entropy of \mathcal{P} is $H(\mathcal{P}) := -\sum_{U \in \mathcal{P}} \mu(U) \log(\mu(U))$. If μ is f -invariant, the (*measure-theoretic*) entropy of f is

$$h_\mu(f) = h_\mu(X, f) := \sup_{\mathcal{P}} \lim_{n \rightarrow \infty} \frac{1}{n} H(\mathcal{P} \vee f^{-1}\mathcal{P} \vee \dots \vee f^{-n}\mathcal{P}),$$

where the supremum is over all finite measurable partitions \mathcal{P} of X , the partition $\mathcal{P} \vee \mathcal{P}'$ consists of all nonempty intersections $U \cap U'$ with $U \in \mathcal{P}$ and $U' \in \mathcal{P}'$, and $f^{-i}\mathcal{P} := \{f^{-i}(U) : U \in \mathcal{P}\}$.

If X is compact, the *topological entropy* of f is

$$h_{\text{top}}(f) = h_{\text{top}}(X, f) = \sup_{\mathcal{U}} \lim_{n \rightarrow \infty} \frac{1}{n} \log N(\mathcal{U} \vee f^{-1}\mathcal{U} \vee \dots \vee f^{-n}\mathcal{U}),$$

where the supremum is over all finite open covers of X , and $N(\mathcal{U})$ is the minimum number of elements of \mathcal{U} needed to cover X . If X is compact and metrizable, the *variational principle* states that $h_{\text{top}}(f) = \sup\{h_\mu(f) : \mu \in M(X, f)\}$, where $M(X, f)$ is the set of all f -invariant Borel probability measures on X . (See, for example, [28, Theorem 8.6].)

Let $\phi \in \mathbb{C}_v(z)$ be a rational function of degree $d \geq 2$, with Julia set $\mathcal{J}_\phi \subseteq \mathbb{P}_{\text{Ber}}^1$, invariant measure $\mu = \mu_\phi$ supported on \mathcal{J}_ϕ , and Jacobian function $\text{Jac}_\phi(x) = d/\deg_x(\phi)$ as in Section 2.2. In [12, Théorèmes C,D], Favre and Rivera-Letelier proved that

$$0 \leq \int_{\mathbb{P}_{\text{Ber}}^1} \log \text{Jac}_\phi(x) d\mu(x) \leq h_\mu(\mathcal{J}_\phi, \phi) \leq h_{\text{top}}(\mathcal{J}_\phi, \phi) = h_{\text{top}}(\mathbb{P}_{\text{Ber}}^1, \phi) \leq \log d.$$

(Actually, if $\text{char } \mathbb{C}_v = p > 0$, then the final d above can be replaced by the sharper $\deg_{\text{sep}}(\phi)$, the smallest degree of $\psi \in \mathbb{C}_v(z)$ such that we can write $\phi(z) = \psi(z^{p^r})$. Since all of our maps will be separable, however, we will always have $\deg_{\text{sep}}(\phi) = d$.)

2.4. Gurevich Entropy. Let \mathcal{A} be a countable set, and let $C = \{c(a, b)\}_{a, b \in \mathcal{A}}$ be an $\mathcal{A} \times \mathcal{A}$ matrix of 1's and 0's. Equip \mathcal{A} with the discrete topology, let $X = \mathcal{A}^{\mathbb{N}}$ equipped with the product topology, and let

$$Y := \{\{a_n\}_{n \geq 0} \in X : c(a_n, a_{n+1}) = 1 \text{ for all } n \geq 0\}$$

be the set of sequences consistent with C . Define the (*one-sided*) *topological Markov shift* $T : Y \rightarrow Y$ by $T(\{a_n\}_{n \geq 0}) = \{a_{n+1}\}_{n \geq 0}$. If \mathcal{A} is finite, then Y is compact, and the variational principle holds for $h_{\text{top}}(Y, T)$. However, if \mathcal{A} is countable, then Y is usually not compact, making it unclear how to even define the topological entropy. Gurevich [14] addressed this issue by defining the *Gurevich entropy* to be

$$h_{\text{Gur}}(T) = h_{\text{Gur}}(Y, T) = \sup h_{\text{top}}(Y', T'),$$

where the supremum is over all subsets $Y' \subseteq Y$ (with associated shift $T' := T|_{Y'}$) formed by restricting to a finite subset $\mathcal{A}' \subseteq \mathcal{A}$ of symbols.

The Gurevich entropy can often be computed combinatorially. For $a, b \in \mathcal{A}$, a *path* of length n from a to b is a finite sequence $a = a_0, a_1, \dots, a_n = b$ with $c(a_{i-1}, a_i) = 1$ for all $i = 1, \dots, n$. For any $n \geq 0$, let $p_{a,b}(n)$ denote the number of paths of length n from a to b . Let R be the radius of convergence (convergence in \mathbb{C} , that is) of the associated generating function $P_{a,b}(z) := \sum_{n \geq 0} p_{a,b}(n)z^n$. If the underlying directed graph is strongly connected — i.e., if for any $a, b \in \mathcal{A}$, there is a path from a to b — then Vere-Jones proved [27] that R is independent of a and b , and Gurevich proved [15] that $h_{\text{Gur}}(T) = -\log R$.

For any $a \in \mathcal{A}$, a *first-return loop* at a is a path from a to a for which none of the intermediate symbols in the path is a . For each $n \geq 1$, let $f_a(n)$ be the number of first-return loops at a of length n . It is easy to show (see, for example, [17, Lemma 7.1.6(iv)] or [25, Equation (1)]) that the associated generating function $F_a(z) := \sum_{n \geq 1} f_a(n)z^n$ satisfies $P_{a,a}(z) = 1/(1 - F_a(z))$ for $|z| < R$. In particular, if $1 - F_a(z)$ has a real root $r > 0$ such that F_a converges and is nonzero on $|z| < r$, then $r = R$, and hence $h_{\text{Gur}}(T) = -\log r$.

For more on Markov shifts on countably many symbols, see, for example, [7] [16, Chapter 1], [17, Chapter 7], [25], or [26].

3. PROOF OF THEOREM A AND RELATED RESULTS

Lemma 3.1. *Let $\phi \in \mathbb{C}_v(z)$ be a nonconstant rational function, and let $\Gamma \subseteq \mathbb{H}_{\text{Ber}}$ be a finite tree. Then $\phi^{-1}(\Gamma)$ is the union of finitely many pairwise disjoint finite trees Z_1, \dots, Z_n . Moreover, for each $i = 1, \dots, n$, we have $\phi(Z_i) = \Gamma$.*

Proof. Pick $y_1, \dots, y_m \in \mathbb{H}_{\text{Ber}}$ such that Γ is the convex hull of $S := \{y_1, \dots, y_m\}$; discarding some of these points if necessary, we may assume that no y_i lies between any other two y_j 's. Let W be the unique connected component of $\mathbb{P}_{\text{Ber}}^1 \setminus S$ intersecting Γ . Note that

W is an open connected affinoid (possibly with finitely many type IV points removed, if some y_i is of type IV), with $\partial W = S$. By [2, Lemma 9.12], $\phi^{-1}(W)$ is a union of at most d pairwise disjoint open connected affinoids U_1, \dots, U_n (again, possibly with finitely many type IV points removed), where $\phi(U_i) = W$ and $\phi(\partial U_i) = S$, for each $i = 1, \dots, n$.

For each i , let $Z_i := \overline{U_i} \cap \phi^{-1}(\Gamma)$. For any $z_1, z_2 \in Z_i$, the image $\phi((z_1, z_2))$ of the open interval strictly between them is a path between $\phi(z_1), \phi(z_2) \in \Gamma$ that never hits S . Thus, $\phi([z_1, z_2]) \subseteq \Gamma$, and hence $[z_1, z_2] \subseteq Z_i$. That is, Z_i is connected. Moreover, since ϕ is an open map, the only points $w \in Z_i$ for which $Z_i \setminus \{w\}$ is still connected are those for which $\phi(w)$ is an endpoint of Γ . In other words, the only endpoints of Z_i belong to $\phi^{-1}(S) \cap \overline{U_i} = \partial U_i$. Since each y_i has at most $\deg \phi$ preimages, ∂U_i is a finite set, and hence Z_i is a finite tree. Finally, because $\phi(\overline{U_i}) = W$, we have $\phi(Z_i) = \Gamma$. \square

Proof of Theorem A. Step 1. First, we will show that the backward orbit $\bigcup_{n \geq 0} \phi^{-n}(x_0)$ intersects I as a dense subset. Let L be the (hyperbolic) length of I , i.e., the $d_{\mathbb{H}}$ -distance between the two endpoints of I . By hypothesis (d), each subinterval $I_i \subseteq I$ has hyperbolic length $c_i^{-1}L \leq L/2$ and contains an element of $\phi^{-b_i}(x_0)$.

Partition each I_i into m subintervals $I_{i,j}^{(2)} := \phi^{-b_i}(I_j) \cap I_i$, for $1 \leq j \leq m$. Each $I_{i,j}^{(2)}$ must have length at most $L/4$ and contain an element of $\phi^{-b_i-b_j}(x_0)$. Continuing in this fashion, we obtain, at the ℓ -th step, a partition of I into m^ℓ subintervals, each of length at most $L/2^\ell$ and containing an element of the backward orbit of x_0 . Letting ℓ increase to infinity, we see that $I \cap \bigcup_{n \geq 0} \phi^{-n}(x_0)$ is dense in I , as claimed.

Step 2. Next, we observe that $\Gamma \subseteq \mathcal{J}_\phi$. Indeed, since $x_0 \in \mathcal{J}_\phi$ by hypothesis (a), and since the Julia set is closed and invariant under ϕ , it follows from Step 1 that $I \subseteq \mathcal{J}_\phi$. By hypothesis (c) and the invariance of \mathcal{J}_ϕ , then, we have $\Gamma \subseteq \mathcal{J}_\phi$.

Step 3. Now we show that \mathcal{J}_ϕ is connected. For every integer $n \geq 0$, let $\Gamma_n := \phi^{-n}(\Gamma)$, and let $X_n := \Gamma_0 \cup \dots \cup \Gamma_n$. Then $X_n \subseteq \mathcal{J}_\phi$, again by the invariance of the Julia set.

We will show, by induction, that X_n is a finite tree. By hypothesis, $X_0 = \Gamma$ is a finite tree. If we know X_n is a finite tree, then by Lemma 3.1, $\phi^{-1}(X_n)$ is a finite union $Z_1 \cup \dots \cup Z_M$ of finite trees, each of which maps onto X_n under ϕ . By hypothesis (b), then, since $x_0 \in \Gamma \subseteq X_n$, each tree Z_i has a nontrivial intersection with Γ , including at least one point of $\phi^{-1}(x_0)$. Hence, $X_{n+1} = X_n \cup \Gamma$ is a finite union of finite trees, each of which intersects $\Gamma \subseteq X_{n+1}$. Since $X_{n+1} \subseteq \mathbb{P}_{\text{Ber}}^1$ has no loops, it is indeed a finite tree.

Thus, $X := \bigcup_{n \geq 0} X_n = \bigcup_{n \geq 0} \Gamma_n \subseteq \mathcal{J}_\phi$ is connected. Since $\bigcup_{n \geq 0} \phi^{-n}(x_0) \subseteq X$ is dense in \mathcal{J}_ϕ , then, $\mathcal{J}_\phi = \overline{X}$ is the closure of a connected set and hence is connected.

Step 4. If Γ is not contained in an interval, there is a point $y \in \Gamma$ such that $\Gamma \setminus \{y\}$ has at least three components. Because $\mathbb{P}_{\text{Ber}}^1$ is uniquely path-connected, y is also a branch point of $\mathcal{J}_\phi \supseteq \Gamma$. By the invariance of \mathcal{J}_ϕ and the mapping properties of residue classes, then, any point x in the backward orbit of y is also a branch point of \mathcal{J}_ϕ . Since $\bigcup_{n \geq 0} \phi^{-n}(y)$ is dense in \mathcal{J}_ϕ , we are done. \square

As promised in the introduction, we can prove in some cases that \mathcal{J}_ϕ has a dense subset of points of *infinite* branching, i.e., points $x \in \mathcal{J}_\phi$ for which $\mathcal{J}_\phi \setminus \{x\}$ has infinitely many connected components. To state the relevant condition, we need the following definition.

Definition 3.2. Let $p := \text{char } k$ be the residue characteristic of \mathbb{C}_v , let $\phi \in \mathbb{C}_v(z)$ be a rational function, and let $x \in \mathbb{P}_{\text{Ber}}^1$ be a point of type II that is periodic of period $m \geq 1$. Let $h \in \text{PGL}(2, \mathbb{C}_v)$ map x to $\zeta(0, 1)$, and let $\psi := h \circ \phi^m \circ h^{-1}$, so that the reduction

$\bar{\psi} \in k(z)$ has degree $\deg \bar{\psi} \geq 1$. If $\bar{\psi}$ is purely inseparable, i.e., if $\bar{\psi}(z) = \eta(z^{p^r})$ for some $\eta \in \text{PGL}(2, k)$ and some $r \geq 0$, we say that x is a *purely inseparable* periodic point of ϕ .

Theorem 3.3. *Let ϕ , Γ , and I be as in Theorem A. Suppose that Γ has a branch point y that is periodic under ϕ . Suppose also that y is not a purely inseparable repelling periodic point. Then \mathcal{J}_ϕ has a dense set of points x at which \mathcal{J}_ϕ has infinite branching.*

Proof. Replacing ϕ by an iterate, we may assume y is fixed. Since y has at least three residue classes, it must be of type II. After a change of coordinates, then, we may assume that $y = \zeta(0, 1)$ is the Gauss point. Let $\bar{\phi} \in k(z)$ denote the reduction of ϕ , so that $\deg(\bar{\phi}) \geq 1$. We claim that $\zeta(0, 1)$ is a point of infinite branching of \mathcal{J}_ϕ .

If $\deg(\bar{\phi}) = 1$, then since $\zeta(0, 1) \in \mathcal{J}_\phi$, there must be a residue class U of $\zeta(0, 1)$ that intersects \mathcal{J}_ϕ but such that the corresponding point u of $\mathbb{P}^1(k)$ is not periodic under $\bar{\phi}$. (See [3, Theorem 6.19] and the discussion that follows it.) The backward orbit $\bigcup_{n \geq 0} \bar{\phi}^{-n}(u)$ in $\mathbb{P}^1(k)$ is therefore infinite, and hence there are infinitely many corresponding residue classes U' of $\zeta(0, 1)$ whose forward orbits contain U . These infinitely many residue classes therefore all intersect \mathcal{J}_ϕ .

On the other hand, if $\deg(\bar{\phi}) \geq 2$, then $\zeta(0, 1)$ is a repelling fixed point; by hypothesis, then, $\bar{\phi}$ is not purely inseparable. Thus, there are at most two points of $\mathbb{P}^1(k)$ that have finite backward orbit under $\bar{\phi}$. (Outside of the purely inseparable case, the Riemann-Hurwitz formula applies, bounding the number of such exceptional points, just as in complex dynamics.) Since there are at least three directions at $\zeta(0, 1)$ intersecting \mathcal{J}_ϕ , there must be at least one such direction U for which the corresponding point $u \in \mathbb{P}^1(k)$ has infinite backward orbit under $\bar{\phi}$. As in the previous paragraph, the infinitely many corresponding preimage residue classes must all intersect \mathcal{J}_ϕ .

Either way, then, infinitely many residue classes at $\zeta(0, 1)$ intersect \mathcal{J}_ϕ . That is, y is a point of infinite branching of \mathcal{J}_ϕ . As in Step 4 of the proof of Theorem A, then, all elements of the backward orbit of y are also points of infinite branching of \mathcal{J}_ϕ . Since $\bigcup_{n \geq 0} \phi^{-n}(y)$ is dense in \mathcal{J}_ϕ , we are done. \square

4. COMPUTING THE ENTROPY

The heart of Theorem B is conclusion (a), that the countable partition \mathcal{P} is a one-sided generator of finite entropy. Being a one-sided generator for ϕ means that the σ -algebra generated by $\{\phi^{-n}(U) : U \in \mathcal{P} \text{ and } n \geq 0\}$ is the full Borel σ -algebra on \mathcal{J}_ϕ . The finiteness of the entropy, meanwhile, is by the following lemma.

Lemma 4.1. *Let $\phi \in \mathbb{C}_v(z)$, μ , and \mathcal{P} be as in Theorem B. Then*

$$\sum_{U \in \mathcal{P}} -\mu(U) \log(\mu(U)) < \infty.$$

Proof. If \mathcal{P} is finite, the statement is trivial. Thus, we may assume that \mathcal{P} is infinite, and hence that \mathcal{J}_ϕ has more than one point. As noted in Section 2.1, \mathcal{J}_ϕ is therefore uncountable. Since each of the countably many elements of \mathcal{P} either is countable or contains a nonempty open set, at least one $U_0 \in \mathcal{P}$ must contain a nonempty open set. By properties of the Julia set, there must be some $N \geq 1$ such that $\phi^N(U_0) = \mathcal{J}_\phi$.

For every integer $n \geq 0$, define

$$\mathcal{P}_n := \{U \in \mathcal{P} : \phi^i(U) \cap U_0 = \emptyset \text{ for all } i = 0, \dots, n-1\},$$

so that $\mathcal{P} = \mathcal{P}_0 \supseteq \mathcal{P}_1 \supseteq \dots$. Next, define $A_n := \bigcup_{U \in \mathcal{P}_n} U$, so that $\mathcal{J}_\phi = A_0 \supseteq A_1 \supseteq \dots$.

Step 1. We claim that $A_{n+N} \subseteq \phi^{-N}(A_n) \cap A_N$ for any $n \geq 0$. Clearly $A_{n+N} \subseteq A_N$. Given $U \in \mathcal{P}_{n+N}$, then, it suffices to show that $\phi^N(U) \subseteq A_n$. By the hypotheses of Theorem B, $\phi^N(U) = \bigcup_{V \in \mathcal{R}} V$ for some subset $\mathcal{R} \subseteq \mathcal{P}$. Since

$$\phi^i(V) \cap U_0 \subseteq \phi^i(\phi^N(U)) \cap U_0 = \phi^{N+i}(U) \cap U_0 = \emptyset$$

for all $V \in \mathcal{R}$ and all $i = 0, \dots, n-1$, we have $\mathcal{R} \subseteq \mathcal{P}_n$. Thus, $\phi^N(U) \subseteq \bigcup_{V \in \mathcal{P}_n} V = A_n$, proving the claim.

Step 2. Next, we claim that $\mu(\phi^{-N}(A_n) \setminus A_N) \geq d^{-N} \mu(A_n)$ for any $n \geq 0$. Since $A_n = \bigcup_{U \in \mathcal{P}_n} U$ is a countable disjoint union, it suffices to show that $\mu(\phi^{-N}(U) \setminus A_N) \geq d^{-N} \mu(U)$ for any $U \in \mathcal{P}_n$. Fix such a U .

By the hypotheses, each iterate $\phi^i(U_0)$ is a union of elements of \mathcal{P} , each of which in turn maps bijectively onto a union of elements of \mathcal{P} . Thus, since $\phi^N(U_0) = \mathcal{J}_\phi$, there is some $V_{N-1} \subseteq \phi^{N-1}(U_0)$ contained in an element of \mathcal{P} such that ϕ maps V_{N-1} bijectively onto U . By similar reasoning, there is some $V_{N-2} \subseteq \phi^{N-2}(U_0)$ contained in an element of \mathcal{P} such that ϕ maps V_{N-2} bijectively onto V_{N-1} . Continuing in this fashion, there is some $V_0 \subseteq U_0$ such that ϕ^N maps V_0 bijectively onto U . Since $N \geq 1$, we have

$$V_0 \subseteq \phi^{-N}(U) \cap U_0 \subseteq \phi^{-N}(U) \setminus A_N.$$

Thus, by properties of the Jacobian discussed in Section 2.2, we have

$$\begin{aligned} \mu(U) &= \mu(\phi^N(V_0)) = \int_{V_0} \text{Jac}_{\phi^N}(x) d\mu(x) = \int_{V_0} \frac{d^N}{\deg_x(\phi^N)} d\mu(x) \\ &\leq \int_{V_0} d^N d\mu(x) = d^N \mu(V_0) \leq d^N \mu(\phi^{-N}(U) \setminus A_N), \end{aligned}$$

proving the claim.

Step 3. By Steps 1 and 2 and the ϕ -invariance of μ , we have

$$\begin{aligned} \mu(A_{n+N}) &\leq \mu(\phi^{-N}(A_n) \cap A_N) = \mu(\phi^{-N}(A_n)) - \mu(\phi^{-N}(A_n) \setminus A_N) \\ &= \mu(A_n) - \mu(\phi^{-N}(A_n) \setminus A_N) \leq (1 - d^{-N}) \mu(A_n) \end{aligned}$$

for any $n \geq 0$. Thus,

$$(4.1) \quad \sum_{n \geq 1} \mu(A_n) \leq \left[\sum_{m \geq 0} (1 - d^{-N})^m \right] \cdot \left[\sum_{i=1}^N \mu(A_i) \right] < \infty.$$

Step 4. For any $n \geq 0$, set $\mathcal{P}'_n := \mathcal{P}_n \setminus \mathcal{P}_{n+1}$. Applying similar reasoning as in Step 2, observe that for any $n \geq 0$ and any $U \in \mathcal{P}'_n$, there is some $V \subseteq U$ such that ϕ^n maps V bijectively onto U_0 . Another computation with the integral of the Jacobian then shows that $\mu(U_0) \leq d^n \mu(V) \leq d^n \mu(U)$. Thus, $\mu(U) \geq d^{-n} \mu(U_0)$.

Moreover, any uncountable $U \in \mathcal{P}$ must contain a nonempty open set, forcing $\phi^n(U) = \mathcal{J}_\phi \supseteq U_0$ for some $n \geq 0$. Hence $U \in \mathcal{P}'_n$. Thus, $\mathcal{P}' := \bigcup_{n \geq 0} \mathcal{P}'_n$ consists of all elements of \mathcal{P} except the countable ones, which have measure zero.

We can now bound the entropy of \mathcal{P} using these observations:

$$\begin{aligned}
h(\mathcal{P}) &= \sum_{U \in \mathcal{P}} -\mu(U) \log(\mu(U)) = \sum_{n \geq 0} \sum_{U \in \mathcal{P}'_n} -\mu(U) \log(\mu(U)) \\
&\leq \sum_{n \geq 0} \sum_{U \in \mathcal{P}'_n} \mu(U) [n \log d - \log(\mu(U_0))] \\
&= -\log \mu(U_0) \sum_{U \in \mathcal{P}'} \mu(U) + \log d \sum_{n \geq 0} \sum_{U \in \mathcal{P}'_n} n \mu(U) \\
&= -\log \mu(U_0) + \log d \sum_{n \geq 1} \mu(A_n) < \infty,
\end{aligned}$$

where the final inequality is by Step 3. \square

Proof of Theorem B. Step 1. Let \mathcal{U} be the set of finite intersections of the form

$$(4.2) \quad U_0 \cap \phi^{-1}(U_1) \cap \cdots \cap \phi^{-n}(U_n),$$

where $n \geq 0$ and $U_0, \dots, U_n \in \mathcal{P}$. We claim that every element of \mathcal{U} is path-connected, proceeding by induction on n . The sets with $n = 0$ are path-connected by hypothesis. If the claim is known for sets with $n = m$, an element of \mathcal{U} with $n = m + 1$ is of the form $U \cap \phi^{-1}(V)$, where $U \in \mathcal{P}$ and V is path-connected. Since $\phi(U)$ and V are path-connected, and since $\mathbb{P}_{\text{Ber}}^1$ is uniquely path-connected, $\phi(U) \cap V$ is path-connected. On the other hand, because ϕ is an open map and, by hypothesis, is injective on U , the restriction $\phi : U \cap \phi^{-1}(V) \rightarrow \phi(U) \cap V$ is a homeomorphism. Thus, $U \cap \phi^{-1}(V)$ is path-connected, proving the claim.

Step 2. By Lemma 4.1, \mathcal{P} has finite entropy. Recall from Section 2.1 that \mathcal{J}_ϕ has a topological basis of countably many open connected affinoids intersected with \mathcal{J}_ϕ . Each open connected affinoid is, in turn, the intersection of finitely many open disks. Thus, to prove statement (a), it suffices to show that for any open disk $D \subseteq \mathbb{P}_{\text{Ber}}^1$, the set $D \cap \mathcal{J}_\phi$ belongs to the σ -algebra generated by the set \mathcal{U} of Step 1.

Give such an open disk D , let $y \in \mathbb{H}$ be its unique boundary point. If $y \notin \mathcal{J}_\phi$, then because \mathcal{J}_ϕ is connected, $D \cap \mathcal{J}_\phi$ either is empty or is all of \mathcal{J}_ϕ ; either way, it is in the σ -algebra generated by \mathcal{U} . Thus, we may assume that $y \in \mathcal{J}_\phi$.

For any $x \in D \cap \mathcal{J}_\phi$, note that $x \neq y$, since $y \notin D$. We claim that for some large enough $n \geq 0$, $\phi^n(x)$ and $\phi^n(y)$ lie in distinct partition elements $U \in \mathcal{P}$. Otherwise, by the hypotheses that some iterate of ϕ is expanding on each U , and that \mathcal{J}_ϕ has finite $d_{\mathbb{H}}$ -diameter, there is some $m \geq 0$ such that $d_{\mathbb{H}}(\phi^m(x), \phi^m(y)) > \text{diam}_{\mathbb{H}}(\mathcal{J}_\phi)$, contradicting the fact that $\phi^m(x), \phi^m(y) \in \mathcal{J}_\phi$.

Hence, there is some $V_x \in \mathcal{U}$ such that $x \in V_x$ but $y \notin V_x$. Since V_x is connected by the claim of Step 1, we must have $x \in V_x \subseteq D \cap \mathcal{J}_\phi$. Thus, $D \cap \mathcal{J}_\phi = \bigcup_{x \in D \cap \mathcal{J}_\phi} V_x$. However, since \mathcal{U} is countable, the union on the right is in fact a countable union. Therefore, $D \cap \mathcal{J}_\phi$ belongs to the σ -algebra on \mathcal{J}_ϕ generated by \mathcal{U} , proving statement (a).

Step 3. Our proof of statement (b) consists mainly of verifying the hypotheses of an entropy formula of Rokhlin. First, as we noted in Section 2.1, \mathcal{J}_ϕ is a separable metrizable space. As a closed subset of $\mathbb{P}_{\text{Ber}}^1$, it is also compact. Thus, by [24, Theorem 15.4.10], it is a Lebesgue space. (This is a technical condition meaning that as a measure space, \mathcal{J}_ϕ is isomorphic to $[0, 1]$ plus perhaps countably many points. See also [28, Theorem 2.1].) Second, recall from Section 2.2 that μ is ergodic on \mathcal{J}_ϕ . Third, \mathcal{P} is a countable one-sided

generator for $\phi : \mathcal{J}_\phi \rightarrow \mathcal{J}_\phi$, and of finite entropy, by statement (a). Fourth, ϕ is essentially countable-to-one in the sense of [19, Definition 2.9.2], since ϕ is finite-to-one.

Thus, by Rokhlin's formula (see, for example, [19, Theorem 2.9.7]), the hypotheses of which we checked in the previous paragraph, and by equation (2.1), we have

$$h_\mu(\phi) = \int_{\mathbb{P}_{\text{Ber}}^1} \log \text{Jac}_\phi(x) d\mu(x) = \int_{\mathbb{P}_{\text{Ber}}^1} \log \left(\frac{d}{\deg_x(\phi)} \right) d\mu(x),$$

proving statement (b).

Step 4. Let Y denote the space of one-sided symbol sequences described in part (c) of Theorem B: the symbol space is the set \mathcal{A} of uncountable elements of \mathcal{P} , and symbol $U \in \mathcal{A}$ can be followed by symbol $V \in \mathcal{A}$ if and only if $V \in S_U$. The topology on Y is inherited from the product topology on $\mathcal{A}^\mathbb{N}$, where we equip the symbol space \mathcal{A} with the discrete topology. Let $T : Y \rightarrow Y$ denote the shift map on Y .

Writing each $U \in \mathcal{P}$ as the union of an open set and a countable set C_U , let $Z_0 \subseteq \mathcal{J}_\phi$ be the (countable) union of all of the countable sets C_U . Then, let $Z := \bigcup_{n \geq 0} \phi^{-n}(Z_0) \subseteq \mathcal{J}_\phi$ be the (countable) set of all points with iterates in Z_0 . Let $J := \mathcal{J}_\phi \setminus Z$.

Define $Q : J \rightarrow Y$ by letting $Q(x)$ be the sequence $\{U_n\}_{n \geq 0}$, where U_n is the unique element of \mathcal{A} containing $\phi^n(x)$. Clearly $Q \circ \phi = T \circ Q$. Moreover, since $J \subseteq \mathbb{P}_{\text{Ber}}^1$ is Hausdorff, part (a) implies that different points in J have different symbol sequences; that is, Q is injective.

Step 5. Let $W := Y \setminus Q(J)$. We claim that W is a countable set.

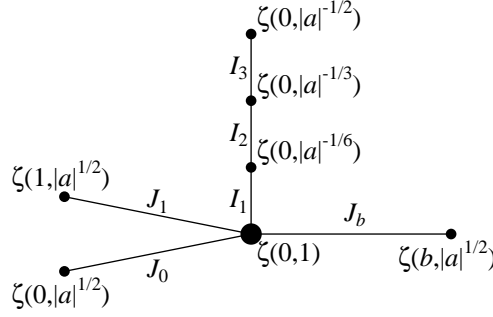
Given $u = \{U_i\}_{i \in \mathbb{N}} \in W$, let $E_n \subseteq U_0$ be the finite intersection given by expression (4.2), for each $n \geq 0$. Clearly $U_0 = E_0 \supseteq E_1 \supseteq E_2 \supseteq \dots$. Define B_u to be the set of all points in $\mathbb{P}_{\text{Ber}}^1$ that are accumulation points of sequences $\{x_n\}_{n \geq 0} \subseteq U_0$ with $x_n \in E_n$.

For each $i \in \mathbb{N}$, the set $\tilde{U}_i := \mathcal{J}_\phi \setminus (Z_0 \cup U_i)$ is a union of opens and hence is open in \mathcal{J}_ϕ . Thus, $\phi^{-i}(\tilde{U}_i)$ is also open. In addition, for any sequence $\{x_n\}_{n \geq 0}$ as in the previous paragraph, we have $x_n \notin \phi^{-i}(\tilde{U}_i)$ for all $n \geq i$. Therefore, $B_u \cap \phi^{-i}(\tilde{U}_i) = \emptyset$, and hence $B_u \subseteq \phi^{-i}(Z_0 \cup U_i)$ for all $i \in \mathbb{N}$. If there were some $x \in B_u \cap \bigcap_{i \in \mathbb{N}} \phi^{-i}(U_i)$, then we would have $u = Q(x)$, contradicting our assumption that $u \in W$. Instead, then, we have $B_u \subseteq \bigcup_{i \in \mathbb{N}} \phi^{-i}(Z_0) = Z$.

We have shown that B_u is a subset of Z for any $u \in W$. For the remainder of this step, fix $z \in Z$, and let $u = \{U_i\}_{i \in \mathbb{N}}$ and $v = \{V_i\}_{i \in \mathbb{N}}$ be two distinct symbol sequences in W with $z \in B_u \cap B_v$. Choose $M \geq 0$ large enough that $\phi^M(z) \in Z_0$, choose $N \geq 0$ large enough that $U_N \neq V_N$, and let $n = \max\{M, N\}$. Define $E_n := \bigcap_{i=0}^n \phi^{-i}(U_i)$ and $F_n := \bigcap_{i=0}^n \phi^{-i}(V_i)$, both of which are connected, by Step 1. In addition, by our choice of n , we have $E_n \cap F_n = \emptyset$; thus, E_n and F_n cannot both intersect a common residue class at z . On the other hand, although one or the other might contain z itself, both E_n and F_n contain points other than z .

Recall from the end of Section 2.1, however, that only countably many residue classes at z can intersect \mathcal{J}_ϕ . Since E_n and F_n are subsets of \mathcal{J}_ϕ , we have shown that there can only be countably many $u \in W$ with $z \in B_u$. Because Z itself is countable, we have proven our claim that W is countable.

Step 6. Define $Y' := Y \setminus W = Q(J)$, so that $Q : J \rightarrow Y'$ is bijective. Y' has a topological basis consisting of cylinder sets, i.e., sets given by specifying the first n symbols $U_0, \dots, U_n \in \mathcal{A}$ of the symbol sequence $u \in Y'$. The inverse image of such a cylinder set under Q is given by expression (4.2) and hence is measurable in J . Thus $Q : J \rightarrow Y'$ is

FIGURE 1. The tree Γ for the map of Section 5.1.

(Borel) measurable. Conversely, by part (a), any Borel subset U of Y' belongs to the σ -algebra generated by \mathcal{U} , and hence $Q(U)$ belongs to the σ -algebra generated by the cylinder sets on Y' , which is the Borel σ -algebra on Y' . Thus, the bijection $Q : J \rightarrow Y'$ gives an isomorphism of (Borel) σ -algebras.

Recalling from Step 4 that $Q \circ \phi = T \circ Q$, then, there is a one-to-one correspondence between the set $M(J, \phi)$ of ϕ -invariant Borel probability measures on J and the set $M(Y', T)$ of T -invariant Borel probability measures on Y' . Moreover, if $\mu \in M(J, \phi)$ and $\nu \in M(Y', T)$ are corresponding such measures, then the entropies $h_\mu(J, \phi)$ and $h_\nu(Y', T)$ clearly coincide.

Meanwhile, since $Z = \mathcal{J}_\phi \setminus J$ is ϕ -invariant, any measure $\mu \in M(J, \phi)$ extends to a measure in $M(\mathcal{J}_\phi, \phi)$ via $\mu(Z) = 0$. Conversely, since Z is countable, any measure $\mu \in M(\mathcal{J}_\phi, \phi)$ either has $\mu(Z) = 1$ and hence has entropy zero, or else it induces a measure $\tilde{\mu} \in M(J, \phi)$, given by $\tilde{\mu}(E) = \mu(E \cap J)/\mu(J)$, of greater entropy. A similar relationship applies to $M(Y, T)$ and $M(Y', T)$, since W is also countable and T -invariant.

Because \mathcal{J}_ϕ is a compact metrizable space, the variational principle applies to it. Meanwhile, Gurevich proved [15] (see also [25, Theorem 1.3]) that $h_{\text{Gur}}(T)$ is the supremum of all the measure-theoretic entropies of $T : Y \rightarrow Y$. Thus,

$$\begin{aligned} h_{\text{top}}(\phi) &= \sup\{h_\mu(\mathcal{J}_\phi, \phi) : \mu \in M(\mathcal{J}_\phi, \phi)\} = \sup\{h_\mu(J, \phi) : \mu \in M(J, \phi)\} \\ &= \sup\{h_\nu(Y', T) : \nu \in M(Y', T)\} = \sup\{h_\nu(Y, T) : \nu \in M(Y, T)\} = h_{\text{Gur}}(T). \quad \square \end{aligned}$$

5. A DEGREE 6 EXAMPLE

5.1. With arbitrary residue field. Fix $a \in \mathbb{C}_v^\times$ with $0 < |a|_v < 1$. Fix $b \in \mathbb{C}_v$ with $|b|_v = |b - 1|_v = 1$, and define

$$\phi(z) = \frac{az^6 + 1}{az^6 + z(z - 1)(z - b)} = 1 + \frac{1 - z(z - 1)(z - b)}{az^6 + z(z - 1)(z - b)}.$$

Let $\Gamma \subseteq \mathbb{P}_{\text{Ber}}^1$ be the convex hull of the four points

$$\zeta(0, |a|_v^{-1/2}), \zeta(0, |a|_v^{1/2}), \zeta(1, |a|_v^{1/2}), \text{ and } \zeta(b, |a|_v^{1/2}).$$

Thus, Γ is a tree consisting of four segments, which we denote I , J_0 , J_1 , and J_b , extending from the Gauss point $\zeta(0, 1)$ to the above four points, respectively. See Figure 5.1.

Break I into the following three equal-length pieces:

$$\begin{aligned} I_1 &= [\zeta(0, 1), \zeta(0, |a|_v^{-1/6})], \quad I_2 = [\zeta(0, |a|_v^{-1/6}), \zeta(0, |a|_v^{-1/3})], \\ \text{and } I_3 &= [\zeta(0, |a|_v^{-1/3}), \zeta(0, |a|_v^{-1/2})]. \end{aligned}$$

Note that the Gauss point $x_0 = \zeta(0, 1) \in I$ is fixed and repelling, since the reduction $\bar{\phi}(z) = 1/(z(z-1)(z-\bar{b}))$ has degree $3 > 1$. In particular, $x_0 \in \mathcal{J}_\phi$. Meanwhile, recalling the definition of the symbol \approx from (1.2), observe that:

- $\phi(\zeta(0, r)) = \zeta(0, 1/r^3)$ for $1 < r < |a|_v^{-1/6}$, since $\phi(z) \approx 1/z^3$ for $1 < |z|_v < |a|_v^{-1/6}$. Thus, ϕ maps I_1 bijectively onto J_0 , stretching $d_{\mathbb{H}}$ by a factor of 3.
- $\phi(\zeta(0, r)) = \zeta(0, |a|_v r^3)$ for $|a|_v^{-1/6} < r < |a|_v^{-1/3}$, since $\phi(z) \approx az^3$ for $|a|_v^{-1/6} < |z|_v < |a|_v^{-1/3}$. Thus, ϕ maps I_2 bijectively onto J_0 , again stretching $d_{\mathbb{H}}$ by a factor of 3, but this time oriented in the opposite direction.
- $\phi(\zeta(0, r)) = \zeta(1, |a|_v^{-1} r^{-3})$ for $|a|_v^{-1/3} < r < |a|_v^{-1/2}$, since $\phi(z) - 1 \approx -1/(az^3)$ for $|a|_v^{-1/3} < |z|_v < |a|_v^{-1/2}$. Thus, ϕ maps I_3 bijectively onto J_1 , stretching $d_{\mathbb{H}}$ by a factor of 3.

In addition, ϕ maps each of J_0, J_1 , and J_b bijectively, and in fact isometrically, onto I .

Therefore, ϕ^2 maps each of I_1, I_2 , and I_3 onto I , stretching distances on each by a factor of 3. In addition, $\Gamma \subseteq I \cup \phi^{-1}(I)$, and ϕ maps $\zeta(0, |a|_v^{-1/3})$ to the Gauss point $x_0 = \zeta(0, 1)$ with local degree 3. Hence, $\phi^{-1}(x_0) = \{x_0, \zeta(0, |a|_v^{-1/3})\} \subseteq \Gamma$, and thus $x_0 \in I \subseteq \Gamma$ satisfies the hypotheses of Theorem A for ϕ . As a result, the Julia set \mathcal{J}_ϕ is connected and has a dense set of branch points. In fact, since $\bar{\phi}(z) = 1/(z(z-1)(z-\bar{b}))$ is separable, those branch points have infinite branching, by Theorem 3.3.

5.2. Intervals of measure zero. Favre and Rivera-Letelier [13] have announced a proof of the following fact: if the invariant measure μ_ψ of a rational function $\psi \in \mathbb{C}_v(z)$ charges an interval in \mathbb{H} , then the Julia set \mathcal{J}_ψ is also contained in an interval. In particular, the interval I contained in our Julia set \mathcal{J}_ϕ of Section 5.1 must have mass $\mu_\phi(I) = 0$.

We can also see this identity directly for our map ϕ , as follows. As we noted, ϕ^2 maps each of I_1, I_2 , and I_3 bijectively onto I , with $\deg_x(\phi^2) = 3$ for all $x \in I$ other than the (finitely many) endpoints of I_1, I_2 , and I_3 . By the Jacobian formula (2.1), for each $i = 1, 2, 3$, we have

$$\mu_\phi(I) = \mu_\phi(\phi^2(I_i)) = \int_{I_i} \text{Jac}_{\phi^2}(x) d\mu_\phi(x) = \int_{I_i} \frac{6^2}{3} d\mu_\phi(x) = 12\mu_\phi(I_i).$$

Thus, since I is the union of I_1, I_2 , and I_3 , any two of which intersect on a set of measure zero, we have

$$\mu_\phi(I) = \mu_\phi(I_1) + \mu_\phi(I_2) + \mu_\phi(I_3) = \frac{1}{4}\mu_\phi(I),$$

proving our claim that $\mu_\phi(I) = 0$.

It follows immediately from this observation that $\mu_\phi(\bigcup_{n \geq 0} \phi^{-n}(I)) = 0$. As noted in the proof of Theorem A, the Julia set \mathcal{J}_ϕ is the closure of $\bigcup_{n \geq 0} \phi^{-n}(I)$. However, viewing \mathcal{J}_ϕ as a tree with infinite branching, this set $\bigcup_{n \geq 0} \phi^{-n}(I)$ includes all of the interior points of the tree. Thus, the set L of leaves — that is, \bar{L} consists of those $x \in \mathcal{J}_\phi$ for which $\mathcal{J}_\phi \setminus \{x\}$ is still connected — has mass $\mu_\phi(L) = 1$.

5.3. The entropy in residue characteristic 3. We will now compute the measure-theoretic and topological entropies of the sextic map ϕ of Section 5.1 when the residue characteristic $\text{char } k$ is 3, when $|3|_v \leq |a|_v < 1$, and when neither of $1, \bar{b} \in \mathbb{P}^1(k)$ is postcritical under the action of $\bar{\phi}(z) = 1/(z(z-1)(z-\bar{b}))$. Since the only critical points of $\bar{\phi}$ are ∞ (with forward orbit $\{0, \infty\}$) and $\gamma := -\bar{b}/(\bar{b}+1)$, this last condition says simply that

1 and \bar{b} are not in the forward orbit of γ . In particular, the condition holds for the map given in the introduction, which has $b = -1$, and hence $\gamma = \infty$.

Claim 5.1. Let $y := \zeta(0, |a|_v^{-1/6})$. Then

$$\phi(y) = \zeta(0, |a|_v^{1/2}), \quad \phi^2(y) = \phi^4(y) = \zeta(0, |a|_v^{-1/2}), \quad \text{and} \quad \phi^3(y) = \zeta(1, |a|_v^{1/2}),$$

with $\deg_y(\phi) = 6$ and $\deg_{\phi^2(y)}(\phi^2) = 3$. Moreover, setting

$$W := \{x \in \mathbb{P}_{\text{Ber}}^1 : |x|_v = |a|_v^{-1/6}\} \setminus \{y\},$$

we have $\phi(W) = \overline{\mathcal{D}}(0, |a|_v^{1/2}) \setminus \{\phi(y)\}$, and

$$\phi^2(W) = \phi^4(W) = \{x \in \mathbb{P}_{\text{Ber}}^1 : |x|_v \geq |a|_v^{-1/2}\} \setminus \{\zeta(0, |a|_v^{-1/2})\}.$$

In particular, $\phi^i(W) \subseteq \mathcal{F}_\phi$ for all $i \geq 0$.

Proof. Let $\psi(z) := a^{-1/2}\phi(a^{-1/6}z)$, so that $\bar{\psi}(z) = (z^6 + 1)/z^3$. Thus, $\deg \bar{\psi} = 6$, and hence we have $\phi(y) = \zeta(0, |a|_v^{1/2})$, with $\deg_y(\phi) = 6$. Similar computations confirm the claimed values of $\phi^i(y)$ for $i = 2, 3, 4$, and that $\deg_{\phi^2(y)}(\phi^2) = 3$.

The characterization of each $\phi^i(W)$ also comes from direct computation. Since $\phi^j(W) = \phi^2(W)$ for $j \geq 2$ even, and $\phi^j(W) = \phi^3(W)$ for $j \geq 3$ odd, the union of the forward images of any $\phi^i(W)$ omits infinitely many points of $\mathbb{P}_{\text{Ber}}^1$, and hence $\phi^i(W) \subseteq \mathcal{F}_\phi$. \square

Define $V_0 := \{\phi(y)\} = \{\zeta(0, |a|_v^{1/2})\}$, $V_\infty := \{\phi^2(y)\} = \{\zeta(0, |a|_v^{-1/2})\}$, and $V_1 := \{\phi^3(y)\} = \{\zeta(1, |a|_v^{1/2})\}$. By Claim 5.1, each of V_0, V_1, V_∞ is an endpoint of \mathcal{J}_ϕ .

The set $\mathcal{J}_\phi \setminus \{\zeta(0, 1)\}$ consists of countably many branches: one extending towards ∞ , and the rest of the form $U_c := \mathcal{J}_\phi \cap \mathcal{D}(c, 1)$ where $c \in k$. (Here, we abuse notation by lifting $c \in k$ to an element of \mathcal{O} , which we also denote c .) Let $\mathcal{C} \subseteq k$ be the set of those $c \in k$ for which $U_c \neq \emptyset$, and let $\mathcal{C}' := \mathcal{C} \setminus \{0, 1\}$. Define $U'_0 := U_0 \setminus V_0$ and $U'_1 := U_1 \setminus V_1$. Meanwhile, let

$$U_{\infty,1} := \{x \in \mathcal{J}_\phi : |x|_v \geq |a|_v^{-1/6}\} \setminus V_\infty, \quad \text{and} \quad U_{\infty,2} := \{x \in \mathcal{J}_\phi : 1 < |x|_v < |a|_v^{-1/6}\}.$$

Finally, let $V = \{\zeta(0, 1)\}$. Clearly,

$$\mathcal{P} := \{V, V_0, V_1, V_\infty, U_{\infty,1}, U_{\infty,2}, U'_0, U'_1\} \cup \{U_c : c \in \mathcal{C}'\}$$

is a countable partition of \mathcal{J}_ϕ , where V, V_0, V_1 , and V_∞ are singletons, $U_{\infty,1}$ is the union of an open set and the singleton $\{y\}$, and the remaining elements of \mathcal{P} are open in \mathcal{J}_ϕ .

We have $\phi(V) = V$, $\phi(V_0) = \phi(V_1) = V_\infty$, and $\phi(V_\infty) = V_1$. In the notation of Section 5.1, $U_{\infty,2} \cap \Gamma$ is the interval I_1 with its endpoints removed, while $U_{\infty,1} \cap \Gamma = I_2 \cup I_3$. Thus, $\phi(U_{\infty,2}) = U'_0$, $\phi(U_{\infty,1}) = \mathcal{J}_\phi$, and

$$\phi(U'_0) = \phi(U'_1) = U_{\infty,1} \cup U_{\infty,2}, \quad \text{while} \quad \phi(U_b) = U_{\infty,1} \cup U_{\infty,2} \cup V_\infty.$$

There are also three values of c (namely, the roots of $\bar{\phi}(c) = 1$) for which $\phi(U_c) = U'_1 \cup V_1$. Finally, $\phi(U_c) = U_{\bar{\phi}(c)}$ for every other $c \in \mathcal{C}' \setminus \{\bar{b}\}$. Note, by our assumed condition on b , that ϕ is injective on U'_0, U'_1 , and each U_c , with local degree 1 at all points of these sets.

Claim 5.2. With notation as above, $\deg_x(\phi) = 3$ for all $x \in U_{\infty,1} \cup U_{\infty,2} \setminus \{y\}$. Moreover, \mathcal{J}_ϕ has finite hyperbolic diameter, and ϕ is injective on both $U_{\infty,1}$ and $U_{\infty,2}$.

Proof. For $\alpha, w \in \mathbb{C}_v$ with $1 < |w|_v < |\alpha|_v < |a|_v^{-1/6}$, we have $\phi(\alpha + w) - \alpha^{-3} \approx -\alpha^{-6}w^3$, and thus $\phi : \overline{D}(\alpha, r) \rightarrow \overline{D}(\alpha^{-3}, |a|_v^{-6}r^3)$ is 3-to-1 for all $1 \leq r \leq |\alpha|_v$. (Here, we are using the hypothesis that $|3|_v \leq |a|_v$.) Hence, $\deg_{\zeta(\alpha, r)}(\phi) = 3$ for all such α, r . In addition, we have $\phi(\overline{D}(\alpha, 1)) = \overline{D}(\alpha^{-3}, |a|_v^{-6})$. Since $\phi(z) \approx 1/bz$ for $z \in D(0, 1)$, it follows that $\phi^2 : \overline{D}(\alpha, 1) \rightarrow \overline{D}(b^{-1}\alpha^3, 1)$.

Similarly, if $|a|_v^{-1/6} < |\alpha|_v < |a|_v^{-1/3}$ and $1 \leq r \leq |\alpha|_v$, then $\phi : \overline{D}(\alpha, r) \rightarrow \overline{D}(a\alpha^3, |a|_v r^3)$ is also 3-to-1, with $\phi(\overline{D}(\alpha, 1)) = \overline{D}(a\alpha^3, |a|_v)$. Likewise, if $|a|_v^{-1/3} < |\alpha|_v < |a|_v^{-1/2}$ and $1 \leq r \leq |\alpha|_v$, then $\phi : \overline{D}(\alpha, r) \rightarrow \overline{D}(1/(1 + a\alpha^3), |a\alpha^6|_v^{-1}r^3)$ is again 3-to-1, with $\phi(\overline{D}(\alpha, 1)) = \overline{D}((a(b-1)\alpha^3)^{-1}, |a|_v)$. In either case, we get $\deg_{\zeta(\alpha, r)}(\phi) = 3$ for all such α, r , and $\phi^2(\overline{D}(\alpha, 1)) \subseteq \overline{D}(\phi^2(\beta, 1))$ with $|\beta|_v > 1$.

If $1 \leq r \leq |\alpha|_v = |a|_v^{-1/3}$, then $\deg_{\zeta(\alpha, r)}(\phi) = 3$, although the point $\phi(\zeta(\alpha, r))$ could lie in $\overline{D}(0, 1)$. In fact, the image $\phi(D)$ of the disk $D = \overline{D}(\alpha, 1)$ is of the form $\overline{D}(\beta, |a\beta^2|_v)$ with $|\beta|_v \geq 1$. If $|\beta|_v \leq |a|_v^{-1/2}$, so that $\phi(D)$ is not contained in $\phi^2(W) \subseteq \mathcal{F}_\phi$, then the radius of $\phi(D)$ is at most 1. In addition, if $|\beta|_v = 1$, then $\phi(D) = \overline{D}(\beta, |a|_v)$.

One consequence of the previous three paragraphs is that for any disk $D = \overline{D}(\alpha, 1)$ with $1 < |\alpha|_v < |a|_v^{-1/2}$, then either $D \subseteq W$, or

- $\phi(D)$ is of the form $\overline{D}(\beta, s)$, where $|\beta|_v \leq 1$ and $s \leq |a|_v$, or
- $\phi(D)$ or $\phi^2(D)$ is of the form $\overline{D}(\beta, s)$, where $|\beta|_v > 1$ and $s \leq 1$.

Similarly, if $D = \overline{D}(\alpha, |a|_v)$ with $|\alpha|_v \leq 1$, we get the same conclusions. All such disks therefore map into one another, and hence they all lie in the Fatou set. Thus, $\mathcal{J}_\phi \setminus \overline{D}(0, 1)$ consists only of points of the form $x = \zeta(\alpha, r)$ with $r \geq 1$. By the same three paragraphs, $\deg_x(\phi) = 3$ for all $x \in U_{\infty, 1} \cup U_{\infty, 2} \setminus \{y\}$, proving the first statement of the claim.

Recall that $U_{\infty, 1} \setminus \{y\}$ maps onto $\mathcal{J}_\phi \setminus \{\phi(y)\}$, and $U_{\infty, 2}$ maps onto U'_0 , while each U_c maps bijectively onto $U_{\overline{\phi}(c)}$. (Or onto $U'_1 \cup V_1$, if $\overline{\phi}(c) = 1$.) Thus, by equation (1.1), ϕ must be injective on both $U_{\infty, 1}$ and $U_{\infty, 2}$. Finally, we have seen that if $\zeta(\alpha, r) \in \mathcal{J}_\phi$, then either $1 < |\alpha|_v \leq |a|_v^{-1/2}$ and $1 \leq r \leq |\alpha|_v$, or else $|a|_v^{1/2} \leq |\alpha|_v \leq 1$ and $|a|_v \leq r \leq |\alpha|_v$. Any such point lies within hyperbolic distance less than $-2 \log |a|_v$ from $\zeta(0, 1)$. Since \mathcal{J}_ϕ is connected, it follows that any point of \mathcal{J}_ϕ (not just those of type II or III) also lies within that distance of $\zeta(0, 1)$. Thus, \mathcal{J}_ϕ has finite hyperbolic diameter, proving the claim. \square

Claim 5.2 confirms most of the hypotheses of Theorem B, including the first bullet point. In addition, because y is the only point in $\{x \in \mathcal{J}_\phi : |x|_v = |a|_v^{-1/6}\}$, the sets $U_{\infty, 1} \setminus \{y\}$ and $U_{\infty, 2}$ are both connected, and $\deg_x(\phi) = 3$ is constant on each. Thus, $d_{\mathbb{H}}(\phi(x), \phi(x')) = 3d_{\mathbb{H}}(x, x')$ for any $x, x' \in U_{\infty, 1}$, and similarly for $U_{\infty, 2}$. Since U'_0 and U'_1 map onto $U_{\infty, 1} \cup U_{\infty, 2}$, while $U_{\overline{b}}$ maps onto $U_{\infty, 1} \cup U_{\infty, 2} \cup V_\infty$, and every other U_c maps onto $U_{\overline{\phi}(c)}$ (and hence eventually onto $U_{\infty, 1} \cup U_{\infty, 2} \cup V_\infty$), we have verified the remaining hypotheses of Theorem B.

Invoking part (b) of the Theorem, then,

$$h_\mu(\phi) = \int_{\mathbb{P}_{\text{Ber}}^1} \log(6/\deg_x(\phi)) d\mu(x) = (\log 6)\mu(\overline{D}(0, 1)) + (\log 2)\mu(U_{\infty, 1} \cup U_{\infty, 2}),$$

where $\mu = \mu_\phi$ is the invariant measure. However, since $\phi : U_{\infty, 1} \rightarrow \mathcal{J}_\phi$ is bijective, with constant local degree $\deg_x(\phi) = 3$ except at the one point $x = y$, the defining property of

Jac_ϕ shows that $\mu(U_{\infty,1}) = 1/2$. Similarly,

$$\mu(U'_0) = \frac{1}{6}\mu(U_{\infty,1} \cup U_{\infty,2}) = \frac{1}{12} + \frac{1}{6}\mu(U_{\infty,2}), \quad \text{and} \quad \mu(U_{\infty,2}) = \frac{1}{2}\mu(U'_0).$$

Thus, $\mu(U_{\infty,2}) = 1/22$, and hence $\mu(U_{\infty,1} \cup U_{\infty,2}) = 6/11$. Substituting into the above formula therefore gives

$$h_\mu(\phi) = \frac{5}{11} \log 6 + \frac{6}{11} \log 2 = \log 2 + \frac{5}{11} \log 3.$$

We now compute the Gurevich entropy of the resulting Markov shift on the set of symbols $\mathcal{A} = \{U_{\infty,1}, U_{\infty,2}, U'_0, U'_1, U_{\bar{b}}, \dots\}$. With that ordering of the symbols, the transition matrix is the transpose of

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

For notational convenience, we will identify each state in \mathcal{A} with the number of its corresponding column; that is, $U_{\infty,1}$ is 1, $U_{\infty,2}$ is 2, U'_0 is 3, U'_1 is 4, $U_{\bar{b}}$ is 5, etc. We will compute $F_1(z)$, the generating function for first-return loops at state 1. As described in Section 2.4, we can compute $h_{\text{Gur}}(T)$, and hence $h_{\text{top}}(\phi)$, from the roots of $1 - F_1$.

To count first-return loops at 1 of various lengths, let us consider four separate types of such loops.

- (a) The unique length one loop 1, 1, giving a contribution of z to $F_1(z)$.
- (b) Loops beginning 1, 2, \dots . The next state after 2 must be 3, followed by either 1 or 2. Thus, we get one first-return loop of every odd length at least 3, of the form 1, 2, 3, 2, 3, \dots , 3, 1. The type (b) loops therefore contribute $z^3 + z^5 + z^7 + \dots = z^3/(1 - z^2)$ to $F_1(z)$.
- (c) Loops beginning 1, a , where $a \in \{3, 4, 5\}$. The next state must be either 1 or 2. Thus, for each of $a = 3, 4, 5$, we get one first-return loop of every positive even length, of the form 1, a , 2, 3, 2, 3, \dots , 3, 1 (where there could be zero copies of 2, 3). The type (c) loops therefore contribute $3(z^2 + z^4 + z^6 + \dots) = 3z^2/(1 - z^2)$ to $F_1(z)$.
- (d) Loops beginning 1, a , where $a \geq 6$. Any such state a is followed by a unique path of some length k through states numbered 6 and higher to either state 4 or 5. Call such a state a a k -state. For each $k \geq 1$, there are 3^k k -states with paths to 4, and 3^k with paths to 5, for a total of $2 \cdot 3^k$ k -states.

After k steps, each loop starting from a k -state looks like the tail of one of type (c), giving a loop of length $k + 2m$ for each $m \geq 1$ and each k -state. Together, then, all $2 \cdot 3^k$ k -states contribute $2 \cdot 3^k z^k (z^2 + z^4 + z^6 + \dots) = 2 \cdot 3^k z^{k+2}/(1 - z^2)$ to $F_1(z)$. Summing across all $k \geq 1$, then, the type (d) loops contribute

$$\frac{6z^3}{(1 - z^2)}(1 + 3z + (3z)^2 + \dots) = \frac{6z^3}{(1 - z^2)(1 - 3z)} \text{ to } F_1(z).$$

Adding up all four types' contributions, we have

$$F_1(z) = z + \frac{z^3 + 3z^2}{(1 - z^2)} + \frac{6z^3}{(1 - z^2)(1 - 3z)} = \frac{z - 3z^3}{(1 - z^2)(1 - 3z)}.$$

Thus,

$$1 - F_1(z) = \frac{(1 - 3z - z^2 + 3z^3) - (z - 3z^3)}{(1 - z^2)(1 - 3z)} = \frac{1 - 4z - z^2 + 6z^3}{(1 - z^2)(1 - 3z)}.$$

The root of $F_1(z)$ of smallest absolute value is $r = 1/\lambda$, where $\lambda \approx 3.8558\dots$ is the largest (real) root of the polynomial $t^3 - 4t^2 - t + 6$. Hence, as claimed in the introduction,

$$h_\mu(\phi) = \log 2 + \frac{5}{11} \log 3 \quad \text{and} \quad h_{\text{top}}(\phi) = \log \lambda.$$

Remark 5.3. The assumption that the residue characteristic $p = \text{char } k$ of \mathbb{C}_v is 3 is essential to the computations of this section. Indeed, if $p \neq 3$, then $\deg_x(\phi) = 1$ for all $x = \zeta(c, r)$ with $1, r < |c|_v$, except possibly for $|c|_v = |a|_v^{-1/6}$. As a result, ϕ is injective and preserves the hyperbolic distance on all of the side branches off $I \setminus \{\zeta(0, |a|_v^{-1/6})\}$. Thus, outside of a certain measure zero set (consisting of those $x \in \mathcal{J}_\phi$ for which all but finitely many iterates $\phi^n(x)$ lie in branches off $|a|_v^{-1/6}$), the points of the leaf set L of Section 5.2 lie at infinite hyperbolic distance from $\zeta(0, 1)$. Hence, μ_ϕ -almost all points of \mathcal{J}_ϕ are of Type I, and therefore $\mu_\phi(\mathbb{H}_{\text{Ber}}) = 0$. By the final statement of [12, Théorème D], then, we have

$$h_\mu(\phi) = h_{\text{top}}(\phi) = \log \deg \phi = \log 6.$$

Thus, $\mu = \mu_\phi$ is a measure of maximal entropy for ϕ if $p \neq 3$.

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